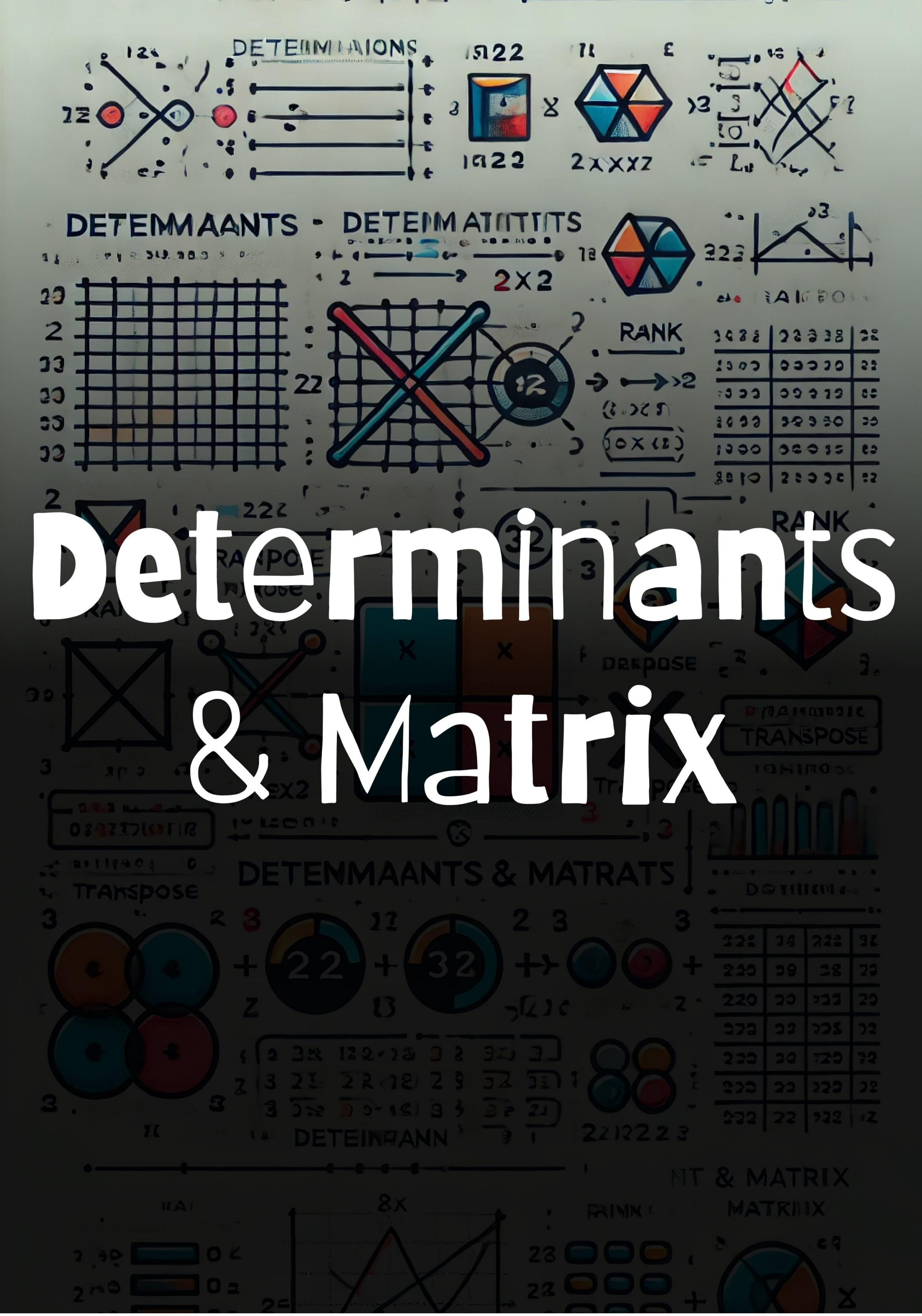


Determinants & Matrix

DETERMINANTS & MATRIX

Symbol, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ → It is called determinant of order 2 and its value is $a_1 b_2 - a_2 b_1$

$|a| = a$ → Determinant of order 1 is the no. itself.

$$\begin{array}{c} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{array}{l} \rightarrow \text{Row } R_1 \\ \rightarrow \text{Row } R_2 \\ \rightarrow \text{Row } R_3 \end{array} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{column } C_1 & \text{column } C_2 & \text{column } C_3 \end{array} \end{array} \Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$D = |a_{ij}|$ i = row, j = column

Value: $a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$= a_1 (b_2 c_3 - c_2 b_3) - b_1 (a_2 c_3 - c_2 a_3) + c_1 (a_2 b_3 - a_3 b_2)$

ARROW METHOD

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

$a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1$

Ques: $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 8 \end{vmatrix}$

Sol:

$= 1(40-48) - 4(16-24) + 7(12-15)$

$= -8 + 32 - 21$

$= 3$

Ques: If $\begin{vmatrix} x & -6 & 1 \\ 2x & -3 & 4 \\ 0 & -1 & 2 \end{vmatrix} = 0$; $x = ?$

Sol:- $x(-6+4) + 6(4x) + 1(-2x) = 0$

$\Rightarrow \quad 22x - 2x = 0$

$\boxed{x = 0}$

Ques: $\begin{vmatrix} K+3 & 1 & -2 \\ 3 & -2 & 1 \\ -K & -3 & 3 \end{vmatrix} = 0$; $K = ?$

Solu:

$(K+3)(-6+3) - 1(9+K) - 2(-9-2K)$

$= -3K - 9 - 9 - K + 18 + 4K = 0$

$\boxed{K \in R}$

NOTE

1. $a_1x + b_1y + c_1 = 0$
   $a_2x + b_2y + c_2 = 0$
   $a_3x + b_3y + c_3 = 0$
   $\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

2. If $(x_1\, y_1), (x_2\, y_2), (x_3\, y_3)$ are three vertices of a Δ

   $$\text{area of } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

3. Equations of lines (concurrent)

   $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

MINORS OF ELEMENT OF A DETERMINANT

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor of $a_{11} = m_{11}$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Similarly, $m_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

CO-FACTOR OF DETERMINANT

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$D = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

$$D = a_{11} C_{11} - a_{12} C_{12} - a_{13} C_{13}$$

$$\Rightarrow \boxed{\sum_{i=1}^{3} a_{1i} C_{1i} = \sum_{i=1}^{3} a_{2i} C_{2i}}$$

Ques:- $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = ?$

sol:- $-a_{11}m_{21} + a_{12}m_{22} - a_{13}m_{23}$

$$\Rightarrow \quad -a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

PROPERTY OF DETERMINANT

(i) If the Row and column are interchanged, then the nature of determinant is unchanged.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$R_1 \leftrightarrow C_1$

★ $D' = -D$ $\qquad D = -D$ if $D' = -0$ it imply that D is called as Skew symmetric determinant 'D' must be equal to zero

Ques: $D = \begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} \Rightarrow D' = \begin{vmatrix} 0 & -b & c \\ b & 0 & -a \\ -c & a & 0 \end{vmatrix} = -D'$

(ii) If any 2 rows and columns of a determinant interchange then the value of determinant is changed by sign only.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{matrix}$$

$\downarrow C_1 \quad \downarrow C_2 \quad \downarrow C_3$

$$D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = -D$$

$R_1 \leftrightarrow R_2$

$$D'' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix} = -D' = D \qquad R_2 \leftrightarrow R_3$$

(iii) If any 2 Rows or columns of a determinant are identical its determinant value is zero.

(iv) If each element of any row and column is equal to zero, the value of determinant will be zero.

(v)
$$\begin{vmatrix} a_1 & b_1 & c_1+\lambda_1 \\ a_2 & b_2 & c_2+\lambda_2 \\ a_3 & b_3 & c_3+\lambda_3 \end{vmatrix} \Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \lambda_1 \\ a_2 & b_2 & \lambda_2 \\ a_3 & b_3 & \lambda_3 \end{vmatrix}$$

If each term of any one row and column can be written as sum of two terms, then the determinant can be written as sum of 2 determinants.

(vi) If by putting $(x=a)$, the value of determinant become zero, then $(x-a)$ is determinant factor

(vii)
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad D' = \begin{vmatrix} Ka_1 & b_1 & c_1 \\ Ka_2 & b_2 & c_2 \\ Ka_3 & b_3 & c_3 \end{vmatrix} = K \cdot D$$

If in each term of any row and/or column is multiplied by some constant K, then the value becomes K times.

(viii)
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad R_1 \to R_1 + R_2 + KR_3$$

$$a_1 = a_2 + Ka_3$$

$$D' = \begin{vmatrix} a_1+a_2+Ka_3 & b_1+b_2+Kb_2 & c_1+c_2+Kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The value of D is not changed by adding to the element any row and column and multiply by corresponding terms of any other row and column.

Ques: Let $D_r = \begin{vmatrix} r & x & \frac{n(n+1)}{2} \\ 2r-1 & y & n^2 \\ 3r-2 & z & \frac{n(3n-1)}{2} \end{vmatrix}$, then find $\sum_{r=1}^{n} D_r = ?$

Sol:- $D_1 + D_2 + D_3 + \ldots + D_n$

$$\sum D_r = \begin{vmatrix} \sum n & x & n(n+1)/2 \\ \sum(2n-1) & y & n^2 \\ \sum(3n-1) & z & n(3n-1)/2 \end{vmatrix}$$

- $2\frac{(n)(n+1)}{2} - n = n^2$
- $\sum r = \frac{n(n+1)}{2}$
- $\sum(3n-1) = \frac{n(3n-1)}{2}$

$\boxed{D=0}$

REMEMBER

- $\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = -(x-y)(y-z)(z-x)$
- $\begin{vmatrix} x^3 & x & 1 \\ y^3 & y & 1 \\ z^3 & z & 1 \end{vmatrix} = -(x-y)(y-z)(z-x)(x+y+z)$
- $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$
- $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

Ques: Find $\begin{vmatrix} 23 & 66 & 11 \\ 36 & 55 & 26 \\ 63 & 143 & 37 \end{vmatrix}$

11 common

Sol:

$$11 \begin{vmatrix} 23 & 6 & 11 \\ 36 & 5 & 26 \\ 63 & 13 & 37 \end{vmatrix}$$

$C_1 \rightarrow C_1 - C_3$

$$= 11 \begin{vmatrix} 12 & 6 & 11 \\ 10 & 5 & 26 \\ 26 & 13 & 37 \end{vmatrix}$$

2 common

$$= \boxed{D = 0}$$

Ques:- Prove that $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (a+c)^2 & b^2 & ac \\ (a+b)^2 & c^2 & ab \end{vmatrix} = -(a^2+b^2+c^2) \times (a+b+c) \times (a-b) \times (b-c) \times (c-a)$

Sol:-

$C_1 \rightarrow C_1 + C_2 - 2C_3$

$$D = \begin{vmatrix} b^2+c^2+a^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ac \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2) \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ac \\ 1 & c^2 & ab \end{vmatrix}$$

Multiply R_1 by a, R_2 by b, R_3 by c

$$D = \frac{a^2+b^2+c^2}{abc} \begin{vmatrix} a & a^3 & abc \\ b & b^3 & abc \\ c & c^3 & abc \end{vmatrix}$$

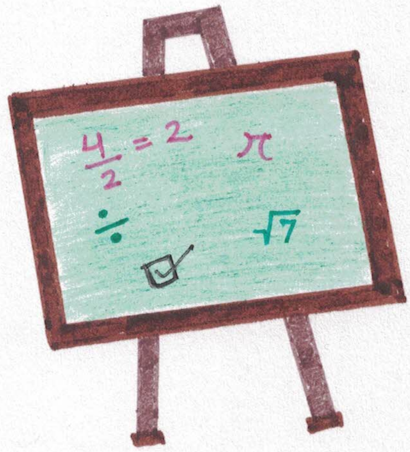
$$D = \frac{(a^2+b^2+c^2)}{abc} \times -(a-b)(b-c)(c-a)$$

$$\Rightarrow \boxed{-(a^2+b^2+c^2) \times (a+b+c) \times (a-b) \times (b-c) \times (c-a)}$$

Ques:- Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

Sol:-

$$D = abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$



$R_1 \longrightarrow R_1 + R_2 + R_3$

$$D = abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \underbrace{\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}}_{1}$$

$$\Rightarrow \boxed{abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Ques:- If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$,

prove that $-\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$

Sol:-

$R_1 \longrightarrow R_1 - R_2$, $R_2 \to R_2 - R_3$

$$D = \begin{vmatrix} p-a & b-q & c-c \\ a-a & q-b & c-r \\ a & b & r \end{vmatrix} = \begin{vmatrix} p-a & -(b+q) & 0 \\ 0 & q-b & -(c+r) \\ a & b & r \end{vmatrix}$$

$$D = (p-a)\left[(q-b)r - b(r-c) - (q-b)\left(-a(r-c)\right]\right.$$

$$\frac{D}{(p-a)(q-b)(r-c)} = \left(\frac{r}{r-c} + \frac{b}{q-b}\right) + \frac{a}{p-a}$$

$$= \frac{r}{r-c} + \frac{b+q-b}{q-b} + \frac{p}{p-a} - 1$$

$$\boxed{\frac{r}{r-c} + \frac{p}{p-a} + \frac{q}{q-b} = 2}$$

Ques: If $(x_1-x_2)^2 + (y_1-y_2)^2 = a^2$

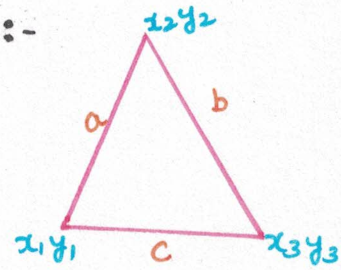
$(x_2-x_3)^2 + (y_2-y_3)^2 = b^2$

$(x_3-x_1)^2 + (y_3-y_1)^2 = c^2$

Prove that,

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Sol:-



$\Rightarrow$ ABC is a Δ

$S = \frac{a+b+c}{2}$ area = 1 $\Delta = \frac{r}{S}$

$$\Delta^2 = \sqrt{(S-a)(S-b)(S-c)3}$$

$$4 \times 4\,\Delta^2 = 4\sqrt{\frac{(-a+b+c)}{2}\,\frac{(a-b+c)}{2}\,\frac{(a+b-c)}{2} \cdot 3}$$

$$4\Delta^2 = \frac{\cancel{4}}{\cancel{4} \times 2}(a-b+c)(a+b-c)\frac{(b+c-a)}{2}\,\frac{(a+b+c)}{2}$$

$$4\Delta^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

PRACTISE

Ques: Let the 3 digit no. A28, 3B9, 62C where a, b, c are integers between 0 and 9 be divisible by K, Prove that,

$$\begin{vmatrix} a & 3 & 6 \\ 8 & 9 & c \\ 2 & B & 2 \end{vmatrix}$$ is also divisible by K.

Ques: If A, B, C are angle of ΔABC, then prove that,

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = 0$$

MULTIPLICATION OF 2 DETERMINANTS

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} \Rightarrow \begin{vmatrix} a_1l_1 + b_1l_2 & a_1m_1 + b_1m_2 \\ a_2l_1 + b_2l_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

Multiplication can be done row to row, column to column or row to column

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$A_1, B_1, C_1 \ldots$ are corresponding co-factor of elements.

PROOF

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix} = D^3$$

$\therefore \quad D \times x = D^3 \Rightarrow x = D^2$

$$\boxed{D^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}}$$

For a determinant of order 'n', determinant of cofactor is D^{n-1}

Example

$$\begin{vmatrix} a_1 l_1 + b_1 m_1 & a_1 l_2 + b_1 m_2 & a_1 l_3 + b_1 m_3 \\ a_2 l_1 + b_2 m_1 & a_2 l_2 + b_2 m_2 & a_2 l_3 + b_2 m_3 \\ a_3 l_1 + b_3 m_1 & a_3 l_2 + b_3 m_2 & a_3 l_3 + b_3 m_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= 0$$

Example Prove that, $\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix}$

$$= 2(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1)$$

Solution: $\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_1) & 1 \\ (a_2-b_1)^2 & (a_2-b_2) & 1 \\ (a_3-b_1)^2 & (a_3-b_3) & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ (a_1-b_2) & (a_2-b_2) & (a_3-b_3) \\ (a_1-b_3)^2 & (a_2-b_2) & (a_3-b_3)^2 \end{vmatrix}$

$a_1^2 - 2a_1 b_1 + b_1^2$

$$\begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & -2b_1 & b_1^2 \\ 1 & -2b_2 & b_2^2 \\ 1 & -2b_3 & b_3^2 \end{vmatrix}$$

$$\Rightarrow \boxed{(a_1-a_2)(a_2-a_3)(a_3-a_1) \times 2(b_1-b_2)(b_2-b_3)(b_3-b_1)}$$

Example Prove that, $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

$= (a^3+b^3+c^3-3abc)^2$

Solution: $R_1 \longrightarrow R_1+R_2+R_3$

$$\Rightarrow \begin{vmatrix} (b+c)^2-a^2 & c^2 & b^2 \\ (a+c)^2-b^2 & 2ac-b^2 & a^2 \\ (a+b)^2-c^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (a^3+b^3+c^3-3abc)^2$$

$$\Rightarrow \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ c & a & -b \\ b & -c & a \end{vmatrix}$$

$$\Rightarrow \boxed{(a^3+b^3+c^3-3abc)\,(a^3+b^3+c^3-3abc)}$$

SYSTEM OF LINEAR EQUATIONS (CRAMER'S RULE)

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

By solving, $x = \frac{D_1}{D}$ $y = \frac{D_2}{D}$ $z = \frac{D_3}{D}$

1. If $D \neq 0$ and atleast one of D_1, D_2, D_3 is not zero, then system of equation is consistent and solution of equation will be unique solution and will be non-trivial

2. If $D \neq 0$ and $D_1 = D_2 = D_3 = 0$, then system of equation is consistent and having a trivial solution.

3. If $D = 0$ and $D_1, D_2, D_3 = 0$, then system of equation is consistent and having infinite non-trival solution.

4. If $D = 0$ and atleast one of D_1, D_2 & D_3 is not equal to zero, then system of equation is inconsistent and have no solution.

HOMOGENEOUS EQUATION

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

$D_1 = D_2 = D_3 = 0$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

1. If $D = 0$, then equation are consistent and having infinite no. of non-trivial solution
2. If $D \neq 0$, then equation are consistent and having a unique trivial solution.

Ques: $\lambda x + y + z = 0$
$-x + \lambda y + z = 0$
$-x - y + \lambda z = 0$

Find λ for non-trivial solution

Sol: $$D = \begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$$= \lambda(\lambda^2 + 1) - 1(-\lambda + 1) + 1(1 + \lambda) = 0$$

$\Rightarrow \lambda^3+\lambda+\lambda^2-1+1+\lambda=0$

$\Rightarrow \lambda(\lambda^2+2+\lambda)=0$

$\boxed{\lambda=0}$

Ques: $x+(\sin\alpha)y+(\cos\alpha)z=0$

$x+(\cos\alpha)y+(\sin\alpha)z=0$

$-x+(\sin\alpha)y-(\cos\alpha)z=0$

Sol:
$$\begin{vmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix}=0$$

$\Rightarrow 1(-\cos^2\alpha-\sin^2\alpha)-\sin\alpha(-\cos\alpha+\sin\alpha)+\cos\alpha(\sin\alpha+\cos\alpha)=0$

$\Rightarrow -1+\sin\alpha\cos\alpha-\sin^2\alpha+\sin\alpha\cos\alpha+\cos^2\alpha=0$

$\Rightarrow 2\sin\alpha\cos\alpha+\cos2\alpha-1=0$

$\Rightarrow \sin2\alpha+\cos2\alpha=1$

$\Rightarrow \sin2\alpha=2\sin^2\alpha$

$\Rightarrow 2\sin\alpha\cos\alpha=2\sin^2\alpha$

$\Rightarrow \sin\alpha(\cos\alpha-\sin\alpha)=0$

$\Rightarrow \boxed{\alpha=n\pi+\frac{\pi}{4}}$

Ques: $(\sin3\theta)x-y+z=0$

$(\cos2\theta)x+4y+3z=0$

$2x+7y+7z=0$

Sol: $\sin3\theta(28-21)-\cos2\theta(-7-7)+2(-3-4)=0$

$\Rightarrow 7\sin3\theta+14\cos2\theta-14=0$

$\sin3\theta+2\cos2\theta-2=0$

$\Rightarrow 3\sin\theta-4\sin^3\theta+2(1-2\sin^2\theta)-2=0$

$\Rightarrow 3\sin\theta-4\sin^3\theta+\cancel{2}-4\sin^2\theta-\cancel{2}=0$

$\Rightarrow 4\sin^3\theta+4\sin^2\theta-3\sin\theta=0$

$\Rightarrow \sin\theta(4\sin^2\theta+4\sin\theta-3)=0$

$\Rightarrow 4\sin^2\theta+4\sin\theta-3=0$

$\Rightarrow 4\sin^2\theta+6\sin\theta-2\sin\theta-3=0$

$\Rightarrow \quad 2\sin\theta(2\sin\theta+3) - 1(2\sin\theta+3) = 0$

$\Rightarrow \quad \sin\theta = \frac{1}{2} \qquad \sin\theta = -\frac{3}{2}$

$\sin\theta = 0$
$\theta = n\pi$

$\Rightarrow \quad \sin\theta = \sin\pi/6$

$\Rightarrow \quad \boxed{\theta = n\pi + (-1)^n \frac{\pi}{6}}$

Ques: $7x - 7y + 5z = 3$
$3x + y + 5z = 7$
$2x + 3y + 5z = 5$

Solu:

$$D = \begin{vmatrix} 7 & -7 & 5 \\ 3 & 1 & 5 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 5 & -10 & 0 \\ 1 & -2 & 0 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= 5(-10+10) = 0$$

$$D_1 = \begin{vmatrix} 3 & -7 & 5 \\ 7 & 1 & 5 \\ 5 & 3 & 5 \end{vmatrix} = \begin{vmatrix} -2 & -10 & 0 \\ 2 & -2 & 0 \\ 5 & 3 & 5 \end{vmatrix}$$

$$= 5(4+20) \neq 0$$

NO SOLUTION

Ques: For what value of p & q, the system of

$2x + py + 6z = 8$
$x + 2y + qz = 5$
$x + y + 3z = 4$

i) No solution
ii) Unique solution
iii) ∞ solution

Soln: Here,

$$\Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix}$$

$$= 2(6-q) - p(3-q) + 6(1-2)$$

$$= 12 - 2q - 3p + pq - 6$$

$$= pq - 2q - 3p + 6 = (p-2)(q-3)$$

Also, $\Delta_1 = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix}$

$= 8(6-q) - p(15-4q) + 6(5-8)$

$= 48 - 8q - 15p + 4pq - 18$

$\Rightarrow -15p + 30 = 4q(p-2)$

$\Rightarrow -15(p-2) = (4q-15)(p-2)$

$\Delta_2 = \begin{vmatrix} 2 & p & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix}$

$= 2(15-4q) - 8(3-q) + 6(4-5) = 0$

$\Delta_3 = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix}$

$= 2(8-5) - p(4-5) + 8(1-2) = p-2$

Case 1 When $\Delta \neq 0$, i.e. $p \neq 2$, $q \neq 3$, given system of equation has a Unique solution

Case 2 When $\Delta = 0$, i.e. $p = 2$ or $q = 3$. When $p = 2$, $\Delta = 0$, $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$. given system of equatio has Infinitely many solutions

Case 3 When $q = 3$, $p \neq 2$, $D = 0$, $D_1 \neq 0$. The system has No solution.

MATRIX

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

OR

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

- A rectangular array of 'm×n' numbers written with bracket is called Matrix
- Unlike determinant, matrix can't be solved
- $A = [a_{ij}]_{m \times n}$ $\quad 1 \le i \le m$, $1 \le i \le n$
- Elements of a matrix may be real or imaginary. All elements real → real matrix

Ques: Construct a matrix of order 3×2 such that $a_{ij} = \left| \frac{i-2j}{2} \right|$

Sol: $A = \begin{bmatrix} 1/2 & 3/2 \\ 1/2 & 1 \\ 1/2 & 1/2 \end{bmatrix} \Rightarrow$ 3×2 Matrix

SPECIAL TYPES OF MATRIX

1. **Row matrix** having only one row

$$A = [a_1 \quad a_2 \; .... \; a_n]_{1\times n}$$

2. **Column matrix** having only one column

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1}$$

3. **Horizontal matrix** A matrix is called horizontal matrix if $A = [a_{ij}]_{m\times n} \rightarrow n > m$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \; ..... & a_{14} \\ a_{21} & a_{22} & a_{23} \; ...... & a_{24} \end{bmatrix}$$

Every column matrix is horizontal matrix

4. **Vertical matrix** A matrix $A = [a_{ij}]_{m\times n}$ is called vertical if $m > n$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

Every column matrix is vertical matrix

5. **Null or Zero matrix** Having all elements equal to zero.

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. **Square matrix**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

No. of rows = No. of column

TYPES OF SQUARE MATRIX

TRIANGULAR MATRICES

(a) Upper triangular matrix

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \qquad a_{ij} = 0, \text{ when } i > j$$

(b) Lower triangular matrix

$$B = \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix} \qquad a_{ij} = 0, \text{ when } i < j$$

★ Min no. of zeros in upper or lower triangular matrix of order 'n' is $\frac{n(n-1)}{2}$

★ Max no. of distinct entries in upper or lower triangular matrix of order n is $\frac{n(n+1)}{2} + 1$

$$\left\{ \underbrace{n^2}_{\text{Total}} - \underbrace{\frac{n(n-1)}{2}}_{\text{Min zeros}} + 1 \rightarrow \text{one distinct entry is zero also} \right\}$$

DIAGONAL MATRIX

$$A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$A = [a_{ij}]$ such that $a_{ij} = 0$, when $i \neq j$

$A = \text{dia}(d_1, d_2, \ldots\ldots d_n)$

Max. no. of zeros in a diagonal matrix $= n - 1$ $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

SCALAR MATRIX

$$A = \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix} \qquad K = \text{Scalar } (K \neq 0)$$

UNIT MATRIX

$$I_3 = A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

★ Min no. of zeros in a unit matrix of order 'n' is $n(n-1)$ $\left\{\frac{n(n-1)}{2} + \frac{n(n-1)}{2}\right\}$

TRACE OF A MATRIX

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Sum of all diagonal elements

$$\boxed{Tr(A) = a_1 + b_2 + c_3}$$

PROPERTIES

- $Tr(A+B) = Tr(A) + Tr(B)$
- $Tr(\lambda A) = \lambda\, Tr(A)$
- $Tr(AB) = Tr(BA)$

Example: $A = \begin{bmatrix} x^2 & 2 & 3 \\ 5 & 2x & 7 \\ -2 & 3 & 24/x \end{bmatrix}$ $f(x) = Tr(A)$. Then find the min. value of $f(x)$ in $[1,5]$

Sol: $f(x) = x^2 + 2x + \frac{24}{x}$

$f'(x) = 2x + 2 - \frac{24}{x^2} = 0$

$x = 2$

$f(3) = 4 + 4 + 32 = 20$

Note:- If,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a_{12} and a_{21}

a_{31} and a_{13} } conjugate element of matrix A

→ In a square matrix $A = [a_{ij}]$ of order n, elements $a_{11}, a_{22}, a_{33} \dots a_{nn}$ are called diagonal or principal diagonal.

→ With every square matrix A, there will be a determinant associated within det(A) or |A|

$$A = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

→ If det |A| = 0, then matrix A is called singular matrix (non- invertible)

→ If $|A| \neq 0$, then matrix A is called non-singular matrix (invertible)

→ Equality of Matrix

$A = [a_{ij}]$ $B = [b_{ij}]$

$n_A = n_B$ (n = order of matrices)

& $a_{ij} = b_{ij}$

→ Addition of 2 matrices

A+B exists if n of A & B are same.

$$A = \begin{bmatrix} a+b & c+d & e+f \\ g+h & i+j & k+l \\ m+n & o+p & q+r \end{bmatrix}$$

$$A = \begin{bmatrix} a & c & e \\ g & i & k \\ m & o & q \end{bmatrix} + \begin{bmatrix} b & d & f \\ h & j & k \\ n & p & r \end{bmatrix}$$

Important:

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) If $B+A = B+A = O$, then O is null matrix and, A and B are called additive inverse of each other.

→ **Multiplication of matrix by a scalar**

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\lambda A = \begin{bmatrix} \lambda a_1 & \lambda b_1 & \lambda c_1 \\ \lambda a_2 & \lambda b_2 & \lambda c_2 \\ \lambda a_3 & \lambda b_3 & \lambda c_3 \end{bmatrix}$$

★ $\boxed{\det [\lambda A]_n = \lambda^n \det(A)}$

If A is square matrix of order n, then -

$\det(\lambda A) = \lambda^n \det(A)$

Question:- If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Find, $2A + 3B - 5I$

Sol:

$$2A + 3B - 5I = \begin{bmatrix} 2+3-5 & 4-6-5 & 0+6-5 \\ -4+6-5 & 2+3-5 & 6+3-5 \\ 0+0-5 & -2+3-5 & 4+0-5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -7 & 1 \\ -3 & 0 & 4 \\ -5 & -4 & -1 \end{bmatrix}$$

ANSWER

Ques: If $2x - y = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$ then $x = ?$, $y = ?$

& $x + 2y = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix}$

Sol: $4x - 2y = \begin{bmatrix} 12 & -12 & 0 \\ -8 & 4 & 2 \end{bmatrix}$

$+ \dfrac{x - 2y}{5x} = \begin{bmatrix} 15 & -10 & 5 \\ -10 & 5 & -5 \end{bmatrix}$

$$\boxed{x = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix}}$$

$$y = \begin{bmatrix} 6 & -4 & 2 \\ -4 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$$

$$\boxed{y = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}}$$

MULTIPLICATION OF 2 MATRICES

1. No. of column in A = no. of rows in B

$A_{3\times \cancel{2}} = B_{\cancel{2}\times 3} \} AB_{3\times 3}$ **Row by Column**

Example

$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} e_1 & e_2 \\ f_1 & f_2 \\ g_1 & g_2 \end{bmatrix}_{3\times 2}$

$$AB = \begin{bmatrix} a_1e_1 + b_1f_1 + c_1g_1 & a_1e_2 + b_1f_2 + c_1g_2 \\ a_2e_1 + b_2f_1 + c_2g_1 & a_2e_2 + b_2f_2 + c_2g_2 \end{bmatrix}_{2\times 2}$$

2. $AB \neq BA$ (In general)

3. If order of A is '$m \times n$' and order of B is '$n \times p$' then $AB = m \times p$

(4) If $AB = BA$, then A and B are called commutative matrix

(5) If $AB = -BA$, then A and B are anti-commutative matrix

(6) If $AB = 0$ (0 is null matrix), it does not imply that $A = 0$ or $B = 0$

(7) If $A = 0$ or $B = 0$ (0 is null matrix $\Rightarrow AB = 0$

(8) $A(BC) = (AB)C$

(9) $A(B+C) = AB + AC$

(10) If $A = \text{dia}(d_1, d_2, \ldots d_n)$

$B = \text{dia}(e_1, e_2, \ldots e_n)$

$AB = \text{dia}(d_1e_1, d_2e_2, \ldots d_ne_n)$

$BA = \text{dia}(e_1d_1, e_2d_2, \ldots e_nd_n)$

$\therefore AB = BA$

(11) If A and B are 2 diagonal matrix or matrix of Same order, then they are commutative

(12) If $A = \text{dia}(A_1, A_2, A_3 \ldots A_n)$, then

$A^2 = \text{dia}(A_1^2, A_2^2, A_3^2 \ldots A_n^2)$

$A^n = \text{dia}(A_1^n, A_2^n, A_3^n \ldots A_n^n)$

$A^n = A \cdot A \cdot A \ldots\ldots$ n times

(13) A^0 is treated as identity matrix of same order

(14) If A is a null matrix, then A^0 is not defined

(15) $A^2, A^3, A^4 \ldots A^n$ are defined when A is a square matrix

(16) $A \cdot I = I \cdot A = A$

(17) $(A+B)^2 = A^2 + AB + BA + B^2$

if $AB = BA$, then

$(A+B)^2 = A^2 + 2AB + B^2$

$(A+B)^3 = A^3 + B^3 + 3A^2B + 3AB^2$

Ques: Matrix A has m rows and n+5 column matrix B has m rows and (11-n) column. If both AB & BA exist. What is m=? & n=?

Sol: $n+5 = m$

$m = 11-n$ $\boxed{m=8}$

$\Rightarrow n+5 = 11-n$

$\Rightarrow 2n = 6$

$\Rightarrow \boxed{n=3}$

Ques: A is a matrix of order n, l = max no. of distinct entries. If A is triangular m = max no. of distinct entries. If A is diagonal, p = min. no. of zeros. If A is triangular.

Sol: $l + 5 = p + 2m$

$$\Rightarrow \frac{n(n+1)}{2} + 6 = \frac{n(n-1)}{2} + 2(n^2 - n^2 - n + 1)$$

$$\Rightarrow \frac{n^2+n+12}{2} = \frac{n^2-n+4n+4}{2}$$

$\Rightarrow \boxed{n=4}$

MATRIX POLYNOMIAL

$P(x) = a_0x^n + a_1x^{n-1} + \dots\dots\dots a_{n-1}x + a_n$

$P(A) = a_0A^n + a_1A^{n-1} + \dots\dots\dots + (a_{n-1})A + a_nI$

$P(A) = 0$ $\Rightarrow$ A is zero of $P(x)$

Null Matrix

A = Square matrix

Ques: $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ $f(x) = x^2 - 4x + 7$

Prove that $f(A) = 0$

Soln: $f(A) = \begin{bmatrix} 4-2 & 6+6 \\ -2-2 & -3+4 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ Hence Proved

VERY SPECIAL TYPE OF MATRIX

IDEMPOTENT MATRICES

$$\boxed{A^2 = A}$$

If A is a square matrix such that $A^2 = A$, then A is called as idempotent matrix.

★ If A is idempotent matrix, then

$A^n = A$ $n \in N$

NILPOTENT MATRICES

$$\boxed{A^K = 0 \quad \begin{cases} K \in N, & 0 \text{ is null matrix} \\ A^{K-1} \neq 0 \end{cases}}$$

A^K is called nilpotent matrix of index or order 'K' if $A^K = 0$

★ eg.- $A^3 = 0$, $A^2 \neq 0$ $\Rightarrow$ A is nilpotent matrix of order 3

PERIODIC MATRIX

$$\boxed{\begin{array}{c} A^{K+1} = A, \quad K \in N \\ \Rightarrow K = \text{period} \end{array}}$$

eg. If $A^5 = A$, then $4 =$ period

NOTE

① If A is idempotent matrix, then it is periodic with period $= 1$

② If A is null matrix, then period is not defined

INVOLUNTARY MATRIX

If, $\boxed{A^2 = I}$, then A is involuntary matrix

Ques: $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, $A^2 = A$?

Sol: $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 4+2-4 & -4-6+4 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$\Rightarrow \boxed{A^2 = A}$ Idempotent Matrix

Ques: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is involuntary $(a, b, c, d \neq 0)$ then find $Tr(A)$.

Sol: $\begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow ac = -dc$ $\quad d^2 + bc = 1$

$\Rightarrow a = -d$

$\Rightarrow \boxed{a + d = 0}$

TRANSPOSE OF A MATRIX

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2\times 3}$$

Row $\rightleftharpoons$ Column

Properties

1) $(A+B)^T = A^T + B^T$
2) $(A^T)^T = A$
3) $K(A)^T = (KA)^T$, $K \to$ scalar
4) $(AB)^T = (B)^T \cdot (A)^T$
5) In general, $(A_1 \cdot A_2 \cdot A_3 \cdot \ldots \ A_n)^T$
$= (A_n)^T \cdot (A_{n-1})^T \ldots \ldots \ A_1^T$

SYMMETRIC & SKEW SYMMETRIC MATRIX

Symmetric: A square matrix $A = [a_{ij}]$ is said to be symmetric matrix if -

$$\boxed{a_{ij} = a_{ji} \qquad \forall\, a_{ij} \in A}$$

$a_{12} = a_{21}$, $a_{31} = a_{13}$, $a_{32} = a_{23}$ or if $\boxed{A^T = A}$

Skew-Symmetric: A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if -

$$\boxed{a_{ij} = -a_{ji} \qquad \forall\, a_{ij} \in A}$$

or if $\boxed{A^T = -A}$

Skew-symmetric matrix:

$a_{12} = -a_{21}$, $a_{31} = -a_{13}$, $a_{32} = -a_{23}$

$a_{11} = -a_{11} \Rightarrow \boxed{a_{11} = a_{22} = a_{33} = 0}$

- For skew symmetric matrix, all its diagonal elements are zero, but converse is not true
- If A is a skew symmetric matrix of odd order, then its determinant, i.e. $\text{Det}(A) = |A| = 0$

Proof: $A^T = -A$

$\Rightarrow \det(A^T) = \det(-A)$

$\Rightarrow \det(A^T) = (-1)^{2n-1} \det(A)$

$\Rightarrow \det A = -\det A$

$\Rightarrow \det(A) = 0$

- If A is a skew symmetric matrix, then its conjugate elements are additive inverse of each other.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Order $n = n + n-1 + n-2 + \dots 1$

$= \frac{(n)(n+1)}{2}$

- If A is a square matrix, then

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = \underbrace{-(A - A^T)}_{\text{skew symmetric}}$$

- If A is a square matrix, then $A + A^T$ will be a symmetric matrix.

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A \Rightarrow \text{Symmetric Matrix}$$

- Any square matrix can be written a sum of symmetric and skew symmetric matrix

$$\boxed{A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)} = \frac{1}{2}(A + A + A^T - A^T) = \frac{1}{2}(2A)$$

ORTHOGONAL MATRIX

$$\boxed{AA^T = I}$$

Ques: If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$

Verify, $[AB]^T = B^T . A^T$

Sol.: $AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}_{3\times1} \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}_{1\times3} = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$

$[AB]^T = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$ —— 1

$(B^T)(A^T) = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \begin{bmatrix} -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$ —— 2

$1 = 2 \rightarrow$ Hence Verified

Ques:- A and B are skew symmetric matrix of same order, then prove that,

a) $A+B$ is symmetric

b) $AB+BA$ is symmetric

c) $AB-BA$ is skew symmetric

Sol: $A^T = -A$ $B^T = -B$

a) $(A+B)^T = A^T + B^T = -A - B = -(A+B)$

$\boxed{(A+B)^T = -(A+B)}$ Skew symmetric

b) $(AB+BA)^T = (AB)^T + (BA)^T$

$= A^T B^T + B^T A^T$

$\boxed{(AB+BA)^T = AB + BA}$ Symmetric

c) $(AB-BA)^T = (AB)^T - (BA)^T$

$= B^T A^T - A^T B^T$

$= BA - AB$

$\boxed{(AB-BA)^T = -(AB-BA)}$ Skew symmetric

PRACTISE QUESTIONS

Ques: If $A = \begin{bmatrix} 1 & -2 \\ 8 & 1 \\ -1 & 2 \end{bmatrix}_{2\times2}$ and $B = \begin{bmatrix} 1 & y-x & -1 \\ x & 1 & z \end{bmatrix}_{2\times3}$

Such that AB is symmetric, then find x & y?

Ques: Express $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as a sum of symmetric & skew symmetric matrices

Ques: If $A = \begin{bmatrix} 5 & 2 & z \\ y & z & -3 \\ 4 & t & -7 \end{bmatrix}$ is a symmetric matrix, then find (x, y, z, t)